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LETTER TO THE EDITOR

Inequivalent quantizations of bi-Hamiltonian systems

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Abstract. *It is shown that the quantization of two Hamiltonians generating (with respect to two different symplectic structures) the same Hamiltonian vector field, may have completely different spectra.*

The purpose of this letter is to point out that the canonical (i.e. geometric) quantization of bi-Hamiltonian systems depends on the Hamiltonian structure. More precisely, we will show that already for the simplest systems two Hamiltonian formulations of the same vector field X may yield (via canonical quantization) completely inequivalent quantum systems (we show examples where generators \hat{X}_0 and \hat{X}_1 of the quantum mechanical time evolution associated to the classical dynamics X have spectra of a completely different type). Here X is a vector field which can be written as a Hamiltonian vector field with respect to a symplectic structure ω_0 and a symplectic structure ω_1 . If we assume that both symplectic structures are quantizable (in the examples we will consider this is trivially true) we can quantize the field X (or its flow) with the ω_0 quantum bundle to obtain \hat{X}_0 or with the ω_1 quantum bundle to obtain \hat{X}_1 . In general their spectra will be completely different. Moreover, as is shown first, this inequivalence arises already at the level of the semiclassical quantization (corrected Bohr-Sommerfeld rules).

To our knowledge this question has been discussed in the literature only by Dodonov *et al* [1], Morandi *et al* [2] and Kaup and Olver [3]. Dodonov *et al* use path integral methods and inequivalent Lagrangians to demonstrate that different quantum systems could result without showing this explicitly in terms of spectra. Working in the Lagrangian formulation they obtain identical second-order equations on the base manifold (=configuration space) from different Lagrangians while the flow on the tangent bundle (=velocity phase space) is different. Morandi *et al* [2] also use the Lagrangian formalism but they do discuss the spectra obtained from quantizing different Lagrangian formulations of the same second-order vector field and show that those can be different. Here we shall be concerned with different Hamiltonian formulations which yield the same vector field (hence, flow) on the cotangent bundle (=momentum phase space). On the other hand Kaup and Olver conjecture in [3] 'that quantization does not depend on the Hamiltonian structure'. Although here we show the contrary it should be noted that the particular example discussed in [3] does indeed yield equivalent quantizations.

For the terminology and results concerning geometric quantization we refer the reader to [4, 5].

We now consider semiclassical quantization of the harmonic oscillator. Let the phase space be $\Gamma = T^*\mathbb{R}$, (q, p) canonical coordinates, $\omega_0 = dp \wedge dq$, $H_0 = \frac{1}{2}(p^2 + q^2)$ and let X defined via

$$\iota(X)\omega_0 = -dH_0 \quad (1)$$

denote the dynamical vector field for the harmonic oscillator. The level sets

$$M_E := \{(q, p) \in T^*\mathbb{R} \mid H_0(q, p) = E\} \quad (2)$$

define for $E > 0$ a regular Lagrangian foliation of $\Gamma \setminus \{p = q = 0\}$ by concentric circles and we have

$$T^*\mathbb{R} \setminus \{p = q = 0\} = \mathbb{R}^2 \setminus \{0\} = S^1 \times \mathbb{R}_+ \quad (3)$$

with the pointwise identification

$$(q, p) \leftrightarrow (\varphi := -\tan^{-1}(p/q), H_0(q, p)). \quad (4)$$

Moreover,

$$\omega_0 = dH_0 \wedge d\varphi \quad (5)$$

and (H_0, φ) are action angle variables for the system (X, ω_0) . We can then obtain the eigenvalues of H_0 from the integrality condition ('semiclassical' quantization which is exact in this case) [4] on the leaves which states that if

$$H_0(q, p) = E_0 := \hbar(n_0 + \frac{1}{2}) \quad n_0 \in \mathbb{Z} \quad (6)$$

has solutions, i.e.

$$H_0^{-1}(E_0) \neq \emptyset \quad (7)$$

then E_0 is an eigenvalue of the quantum operator \hat{H}_0 associated with the classical observable H_0 . Obviously, in our case $H_0^{-1}(E_0) = M_{E_0} \neq \emptyset$ for $n_0 \geq 0$ in (6) and we have the well known spectrum for the harmonic oscillator. Note that in (6) we have already included the Maslov correction $\hbar/2$ which is non-trivial in the case at hand.

Now let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(x) > 0 \forall x \geq 0$ and let $F(x) := \int_0^x f$ (i.e. $dF/dx = f$ and $F(0) = 0$). Then

$$\omega_1 := f \circ H_0(q, p) dp \wedge dq \quad (8)$$

is a symplectic form on Γ and

$$H_1(q, p) := F \circ H_0(q, p) \quad (9)$$

is a Hamiltonian for X , i.e.

$$\iota(X)\omega_1 = -dH_1 \quad (10)$$

moreover

$$\omega_1 = dH_1 \wedge d\varphi \quad (11)$$

and thus (H_1, φ) are action angle variables for the system (X, ω_1) . As before we have that if

$$H_1(q, p) = E_1 := \hbar(n_1 + \frac{1}{2}) \quad n_1 \in \mathbb{Z} \quad (12)$$

has solutions then E_1 is an eigenvalue of the quantum operator \hat{H}_1 associated with H_1 . In (12) we have assumed that the Maslov correction is the same for our ω_0 and

ω_1 [6]. At first glance (12) seems to indicate that both Hamiltonian formulations give the same spectrum for their Hamiltonian functions generating the same classical dynamics X . However, given an eigenvalue $\hbar(n + \frac{1}{2})$ of \hat{H}_0 it may be that

$$H_1^{-1}(\hbar(n + \frac{1}{2})) = \emptyset \tag{13}$$

and thus $\hbar(n + \frac{1}{2}) \notin \text{spec}(\hat{H}_1)$. As an example we might take

$$F(H_0(q, p)) = \frac{\hbar}{4} \left(1 - \exp \left(-\frac{4H_0(q, p)}{\hbar} \right) \right). \tag{14}$$

Then it is easy to see that

$$H_1^{-1}(\hbar(n + \frac{1}{2})) = \emptyset \quad \forall n \in \mathbb{Z} \tag{15}$$

and thus $\text{spec}(\hat{H}_1)$ being empty is completely different from $\text{spec}(\hat{H}_0)$. Choosing a different H_1 appropriately, we may, for example, obtain spectra consisting of a non-zero but finite number of points. Of course, there is a simple relation between the volume given by ω_1 and the function F :

$$\int_{\Gamma} \omega_1 = 2\pi \lim_{\tau \rightarrow \infty} F \left(\frac{\tau^2}{2} \right).$$

Hence, a bounded F (as in (14)) always implies a finite volume, which in turn implies a finite-dimensional quantum Hilbert space. Conversely, if one chooses F such that it is not bounded from above then (since we always assume $F(0) = 0, F' > 0$) the semiclassical quantization yields identical spectra for the quantum Hamiltonians.

Next we consider geometric quantization. As before we consider $\Gamma = T^*\mathbb{R}, \omega_0 = dp \wedge dq$ but now with (so far) arbitrary H_0 and X given as in (1). We choose the vertical polarization

$$P := \text{span}_{\mathbb{R}} \{ \partial / \partial p \} \tag{16}$$

and the potential

$$\theta_0 = p \, dq \tag{17}$$

adapted (i.e. $\theta_0|_P = 0$) to the given polarization. The quantum bundle here is trivial and the Hilbert space \mathcal{H}_0 is given by

$$\mathcal{H}_0 := \left\{ \psi : \Gamma \rightarrow \mathbb{C} \mid \frac{\partial \psi}{\partial p} = 0, \int_{\mathbb{R}} \bar{\psi} \psi \, dq < \infty \right\}. \tag{18}$$

Only functions which leave the polarization invariant can be quantized and those are of the form

$$h(q, p) = a(q)p + b(q). \tag{19}$$

Hence, let

$$H_0(q, p) = a_0(q)p + b_0(q). \tag{20}$$

The vector field X is then quantized by the ω_0 -quantization to the operator [4, 5]

$$\hat{X}_0 = -i\hbar a_0(q) \frac{\partial}{\partial q} + b_0(q) - \frac{i\hbar}{2} \frac{\partial a_0}{\partial q}(q). \tag{21}$$

Let ω_1 and H_1 be defined as in (8) and (9). Then the quantum bundle arising from ω_1 is trivial as well and

$$\theta_1 := G(q, p) \, dq \quad (22)$$

where

$$G(q, p) := \int_0^p f \circ H_0(q, \mu) \, d\mu \quad (23)$$

is a potential adapted to the polarization. Hence

$$\mathcal{H}_1 = \mathcal{H}_0 =: \mathcal{H} \quad (24)$$

i.e. the Hilbert spaces are identical. Although it can easily be done it is not of interest here to determine the ω_1 -quantizable functions. For our purpose it is sufficient to note that $H_1 = F \circ H_0$ is ω_1 quantizable since H_1 generates (with ω_1) X and this vector field leaves the polarization P invariant (as a consequence of the choice of H_0 in (20)). Before we write down the ω_1 -quantized operator \hat{X}_1 corresponding to X we note that

$$\begin{aligned} \Delta(q, p) &:= H_1(q, p) - \theta_1(X) \\ &= \int_0^{a_0(q)p + b_0(q)} f(\mu) \, d\mu - a_0(q) \int_0^p f \circ H_0(q, \mu) \, d\mu \end{aligned} \quad (25)$$

does not depend on p and thus

$$\Delta(q, p) = \Delta(q, 0) = F \circ b_0(q). \quad (26)$$

Consequently, we obtain for the ω_1 quantization of X

$$\hat{X}_1 = -i\hbar a_0(q) \frac{\partial}{\partial q} + F \circ b_0(q) - \frac{i\hbar}{2} \frac{\partial a_0}{\partial q}(q). \quad (27)$$

Suppose we have $H_0 = b(q)$, then

$$X = -\frac{\partial b_0}{\partial q} \frac{\partial}{\partial p} \in P \quad (28)$$

and X is quantized in both the ω_0 and ω_1 quantization as multiplication operator

$$\hat{X}_0 = b_0(q) = H_0 \quad (29)$$

$$\hat{X}_1 = F \circ b_0(q) = H_1. \quad (30)$$

If $\psi \in \mathcal{H}$ is to be an eigenvector of \hat{X}_0 with eigenvalue λ^0 , then $\forall q \in \mathbb{R}$

$$(\lambda^0 - b_0(q))\psi(q) = 0. \quad (31)$$

Hence, we must have

$$\lambda^0 \in b_0(\mathbb{R}) \quad (32)$$

and this is also sufficient for λ^0 to be an eigenvalue. ψ will be an appropriately defined distribution. A similar statement holds for $H_1 = F \circ H_0$. Consequently, if $\lambda \in H_0(\mathbb{R})$ but $\lambda \notin H_1(\mathbb{R})$ then $\text{spec}(\hat{X}_0) \neq \text{spec}(\hat{X}_1)$. For example, we may choose $H_0(q, p) = q$ which implies $\text{spec}(\hat{X}_0) = \mathbb{R}$ and choosing $F(x) = 1/(\exp(-x) + 1)$ we have $\text{spec}(\hat{X}_1) = [0, 1]$. This proves the asserted inequivalence.

The information contained in the Bohr-Sommerfeld rules has to be interpreted as follows. They are conditions on the leaves of the Lagrangian foliation which select those leaves on which the generalized wavefunctions have their support. If no leaves which satisfy the rules exist the Hilbert space obtained from this foliation consists only of the zero vector [4]. Those leaves on which the rules can be satisfied provide support for a non-trivial Hilbert space of generalized functions. The spectrum of the operators corresponding to the action variables is then given by those (constant) values which the action variables take on the leaves satisfying the rules. For completely integrable systems for which the integrals define a Lagrangian foliation of almost all phase space and where all leaves are equivalent to tori the Bohr-Sommerfeld rules give the exact spectra for the action variables. In this sense the semiclassical quantization of the harmonic oscillator discussed above is exact and provides a prototypical example.

In the previous examples we have chosen second symplectic structures and Hamiltonians such that the images of the phase space Γ under H_0 and H_1 differ. In the 'semiclassical' treatment one may have that for H_1 no leaves exist which satisfy the Bohr-Sommerfeld rules in which case the two quantizations cannot be equivalent. On the other hand if for each leaf which maps to $\hbar(n + \frac{1}{2})$ under H_0 a (possibly) different leaf exists which maps to the same value under H_1 then the two semiclassical quantizations are equivalent.

A similar statement holds for the full geometric quantization. Any functions H_0 and H_1 which generate via ω_0 and ω_1 a vector field X which lies in the polarization is quantized as multiplication operator (in the representation chosen by the polarization). Thus, the images of the phase space under the functions H_0 and H_1 determine the spectra of \hat{X}_0 and \hat{X}_1 . So the two quantizations are inequivalent whenever those images differ. On the other hand the two quantizations yield the same spectra whenever those images are identical (as is the case in the example discussed by Kaup and Olver [3]).

These statements also apply to n -dimensional separable systems where one can apply the above analysis on each pair of action-angle variables.

Here we have analysed equivalence in the weakest possible sense, i.e. we have only compared spectra. On physical grounds an equivalence between quantizations would require more than that, e.g. a unitary map intertwining the two quantizations.

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